The Meissner effect and the vortex structure in stacked junctions and layered superconductors: Exact analytical results

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We present an exact mathematical description of the Meissner effect and of the vortex state in periodic thin-layer superconductor/insulator structures with an arbitrary number of identical junctions $N-1$ ($2 \leq N < \infty$, where N is the number of superconducting layers) in the presence of a static parallel external field H. Based on an analytical analysis of the coupled static sine-Gordon (SG) equations for the phase differences, we obtain a complete classification of all possible types of physical solutions. We prove that at $H > 0$ these equations admit only Meissner solutions and topological "vortex-plane" solutions. Both the types of solutions are characterized, in general, by $\left[\frac{N}{2}\right]$ Josephson lengths ($\left[\frac{N}{2}\right]$ is the integer part of $\frac{N}{2}$). We derive an explicit analytical expression for the superheating field of the Meissner state, H_s, as a function of N and show that H_s simultaneously determines the penetration field for the vortex planes. For $H \ll H_s$, we obtain a closed-form analytical expression for the Meissner field and investigate a transition to the infinite ($N = \infty$) layered-superconductor limit. Thermodynamically stable "vortex-plane" solutions represent coherent chains of Josephson vortices (one vortex per each insulating layer in a chain). Being a natural generalization of ordinary Josephson vortices in a single junction, the vortex-plane solutions inherit such properties of the former as periodicity along the layers and the overlapping of states with different topological numbers. We obtain exact analytical expressions for the self-energy of a vortex plane and for the lower critical field H_{c1}. In contrast to a prevailing view, the coupled SG equations do not possess any single-vortex solutions for $H > 0$, as well as such vortex solutions as the "triangular lattice" and the "row mode". Thus, single-vortex solutions appear only in the limit $H = 0$, for which case we provide a detailed analytical description. The general consideration is illustrated by two exactly-solvable examples ($N = 2, 3$). Experimental implications of the results are discussed.

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I. INTRODUCTION

We present a rigorous mathematical examination of the problem of the Meissner effect and of the vortex structure in thin-layer Josephson-junction stacks and layered superconductors, with a static, homogeneous external magnetic field $H > 0$ applied parallel to the layers (along the z axis, see Fig. 1.) We consider periodic systems composed of an arbitrary number $N-1$ of identical superconductor/insulator (S/I) junctions ($2 \leq N < \infty$, where N is the number of S-layers, with x being the layering axis), occupying an arbitrary region $[-L, L]$ along the y axis.

We begin with the investigation of analytical properties of a finite set of coupled static sine-Gordon (SG) equations for phase differences $\phi_n (1 \leq n \leq N-1)$, obtained by the minimization of a microscopic Gibbs free-energy functional.[1,2] Equations of this type were first derived within the framework of a phenomenological approach.[3] Under corresponding redefinition of the parameters, they also apply to the phenomenological Lawrence-Doniach (LD) model[4] with $N < \infty$. Although particular numerical solutions to these equations were obtained in several publications,[3,5–7] any analytical analysis of their general properties has not been undertaken up until now.

Making use of standard methods of the theory of differential equations,[8] we arrive at a complete classification of the solutions to the SG equations subject to appropriate physical boundary conditions at $y = \pm L$. In particular, we prove that for $H > 0$ these equations admit only Meissner solutions and topological "vortex-plane" solutions.[1,2] Both the types of solutions are characterized, in general, by $\left[\frac{N}{2}\right]$ Josephson lengths λ_{Jk} ($\left[\frac{N}{2}\right]$ is the integer part of $\frac{N}{2}$, $k = 0, 1, \ldots, \left[\frac{N}{2}\right]-1$). The Meissner solutions persist up to a certain superheating field of the Meissner state, H_{sL}. We show that the field H_{sL} simultaneously determines the penetration field for the vortex planes. For $L \gg \lambda_{J\max} \equiv \lambda_{J0}$, we derive an explicit analytical expression for the superheating (penetration) field $H_s \equiv H_{s\infty}$ as a function of $N \geq 2$. For $H \ll H_s$, we obtain a closed-form analytical expression for the Meissner local field $H_n(y)$, valid for any $N \geq 2$, and investigate a transition to the infinite ($N = \infty$) layered-superconductor limit.

Thermodynamically stable vortex-plane solutions, physically, represent coherent chains of Josephson vortices (one vortex per each I-layer in a chain), positioned in planes parallel to the coordinate axes x, z. (See Fig. 2.) Such solutions

were thoroughly studied for infinite layered superconductors in Refs.[1,2] . For $H_{c1} \ll H \ll H_{c2}$, they correspond to the "transparent state" considered by Theodorakis within the framework of the infinite LD model.[9] In the case of a double-junction stack ($N = 3$), they coincide with the well-known "in-phase mode".[3,10,11,5] Significantly, coherent Josephson vortex configurations that can be identified with the vortex-plane solutions have been directly observed in experiments on artificial double-junction stacks[12] and weakly-coupled multilayers.[13] Besides giving a proof of the existence and stability of the vortex-plane solutions for $3 \leq N < \infty$, we discuss their main physical properties, such as periodicity in the y direction and the overlapping of states with different numbers of vortex planes N_v. In the limit $L \gg \lambda_{J\,\max} \equiv \lambda_{J0}$, we derive exact analytical expressions for the self-energy E_v of the state with $N_v = 1$ and for the lower critical field H_{c1}. We also investigate a physical and mathematical relationship between vortex-plane solutions for $N \geq 3$ and the ordinary Josephson vortices in a single junction ($N = 2$).

In contrast to a prevailing view, the coupled SG equations do not possess any single-vortex solutions for $H > 0$ and $L < \infty$, as well as such vortex solutions as the "triangular lattice"[14–16] and the "row mode".[6] Thus, single-vortex solutions appear only in the limit $H = 0$, $L = \infty$, which, physically, signifies their absolute thermodynamic instability. In this regard, it should be emphasized that single-vortex configurations were originally proposed for infinite layered superconductors without an appropriate analysis of the actual conditions of their existence.[17,18,15,19] (See Refs.[20,1,2] for the criticism.) An excessive preoccupation with single-vortex configurations in previous publications can be explained by the confusion of the problem of finding the lowest-energy topological solutions at $H = 0$ with the problem of the minimization of the Gibbs free energy at $H > 0$: Although at $H = 0$, $L = \infty$ the energy of single-vortex configurations is lower than that of the vortex planes, the former ones are irrelevant to the problem of minimization because of their absolute instability at $H > 0$, $L < \infty$. For the same reason, single-vortex solutions for $H = 0$, $L = \infty$ cannot be used for estimates of the lower critical field H_{c1}. The absence of single-vortex configurations at $H > 0$ was established for infinite layered superconductors by the use of exact variational methods in Refs.[1,2] . In order to clarify this issue for $N < \infty$, we analyze those features of single-vortex solutions at $H = 0$, $L = \infty$ which preclude their existence at $H > 0$, $L < \infty$.

Section II of the paper is devoted to exact mathematical formulation of the problem. In section III, we derive all major mathematical and physical results sketched above. Mathematical **Lemmas 1**, **2** and the **Theorem** of subsection IIIA form the basis for a subsequent physical analysis in subsections IIIB-IIID. The general consideration of section III is illustrated in section IV by two exactly-solvable examples, namely of a single thin-layer junction ($N = 2$) and of a double-junction stack ($N = 3$). (In section IV, we present a new exact solution to the single SG equation, valid for arbitrary $H \geq 0$ and $L > 0$.) The obtained results are discussed in section V. Appendices A-C contain mathematical derivations and proofs omitted in the main text.

II. FORMULATION OF THE PROBLEM

We consider a periodic structure consisting of alternating N superconducting(S) and $N - 1$ insulating (I) layers ($2 \leq N < \infty$) in a parallel, static, homogeneous external magnetic field $0 < H \ll H_{c2}$, where H_{c2} is the upper critical field. (See Fig. 1.) The S-layer thickness is a, and p is the period; the length of the structure in the y direction is $W = 2L$, and W_z is the length of the structure in the z direction ($W_z \to \infty$). We set $\hbar = c = 1$ and assume that

$$\frac{T_{c0} - T}{T_{c0}} \ll 1, \tag{2.1}$$

$$\xi_0 \ll a, \tag{2.2}$$

$$a \ll \min\left\{\zeta(T), \lambda(T), \alpha^{-1}\xi_0\right\}, \qquad \alpha \equiv \frac{3\pi^2}{7\zeta(3)} \int_0^1 dt t D(t) \ll 1, \tag{2.3}$$

$$a \ll p. \tag{2.4}$$

Here T_{c0} is the critical temperature of an isolated S-layer, ξ_0 is the BCS coherence length; $\zeta(T)$ and $\lambda(T)$ are, respectively, the Ginzburg-Landau (GL) coherence length and the penetration depth; $D(\cos\theta)$ is the incidence-angle-dependent tunneling probability of the I-layer between two successive S-layers.

Conditions (2.1) and (2.2) ensure the applicability of the GL-type expansion within each S-layer. Condition (2.3) corresponds to the thin S-layer limit, whereas condition (2.4) is employed here for the sake of mathematical simplicity only. To simplify further the mathematical description, we introduce dimensionless units by

$$\frac{x}{p} \to x,$$

$$\frac{y}{\bar{\lambda}_J} \to y,$$

$$\frac{H}{\bar{H}_s} \to H,$$

where the quantities on the left-hand side are dimensional, with $\bar{\lambda}_J = (8\pi e j_0 p)^{-1/2}$ being the Josephson penetration depth (j_0 is the density of the Josephson current in a single junction with thick electrodes) and $\bar{H}_s = \left(ep\bar{\lambda}_J\right)^{-1}$ being the superheating (penetration) field of the infinite layered superconductor.[1,2] In our dimensionless units, for example, the flux quantum is $\Phi_0 = \pi$, and the lower critical field of the infinite layered superconductor[1,2] is $\bar{H}_{c1} = \frac{2}{\pi}$.

Under the above conditions, the structure is completely described by a closed set of self-consistent microscopic equations for the reduced modulus of the superconducting order parameter in the nth S-layer $f_n(y)$ ($0 \le f_n \le 1$, $n = 0, 1, \ldots, N-1$) and the phase difference between the nth and $(n-1)$th S-layers $\phi_n(y)$ ($\phi_n \equiv \varphi_n - \varphi_{n-1}$, $n = 1, 2, \ldots, N-1$):[21]

$$f_0(y) - f_0^3(y) = r(T)\left[\frac{\epsilon^2}{2}\frac{d^2 f_0(y)}{dy^2} + \frac{2\left[H - H_1(y)\right]^2}{\epsilon^2 f_0^3(y)} + \frac{1}{2}\left[f_0(y) - f_1(y)\cos\phi_1(y)\right]\right],$$

$$f_n(y) - f_n^3(y) = r(T)\left[\frac{\epsilon^2}{2}\frac{d^2 f_n(y)}{dy^2} + \frac{2\left[H_n(y) - H_{n+1}(y)\right]^2}{\epsilon^2 f_n^3(y)}\right.$$

$$\left.+\frac{1}{2}\left[2f_n(y) - f_{n+1}(y)\cos\phi_{n+1}(y) - f_{n-1}(y)\cos\phi_n(y)\right]\right], \quad 1 \le n \le N-2, \tag{2.5}$$

$$f_{N-1}(y) - f_{N-1}^3(y) = r(T)\left[\frac{\epsilon^2}{2}\frac{d^2 f_{N-1}(y)}{dy^2} + \frac{2\left[H - H_{N-1}(y)\right]^2}{\epsilon^2 f_{N-1}^3(y)}\right.$$

$$\left.+\frac{1}{2}\left[f_{N-1}(y) - f_{N-2}(y)\cos\phi_{N-1}(y)\right]\right];$$

$$\frac{df_n}{dy}(\pm L) = 0, \quad 0 \le n \le N-1; \tag{2.6}$$

$$\frac{1}{f_n^2(y)}\left[H_{n+1}(y) - H_n(y)\right] - \frac{1}{f_{n-1}^2(y)}\left[H_n(y) - H_{n-1}(y)\right] - \epsilon^2 H_n(y)$$

$$= -\frac{\epsilon^2}{2}\frac{d\phi_n(y)}{dy}, \quad 1 \le n \le N-1, \tag{2.7}$$

$$H_0(y) = H_N(y) = H, \tag{2.8}$$

$$\frac{d\phi_n}{dy}(\pm L) = 2H, \quad 1 \le n \le N-1, \tag{2.9}$$

where

$$r(T) \equiv \frac{\zeta^2(T)\alpha}{a\xi_0}, \qquad \epsilon \equiv \frac{\sqrt{ap}}{\lambda},$$

and the local magnetic field in the nth I-layer ($n-1 < x < n$) is given by

$$H_n(y) = \frac{1}{2}\int_{-L}^{y} du f_n(u) f_{n-1}(u) \sin\phi_n(u) + H$$

$$= \frac{1}{2}\int_{L}^{y} du f_n(u) f_{n-1}(u) \sin\phi_n(u) + H. \tag{2.10}$$

Note that although in most physical situations $\epsilon \ll 1$, we will not need the smallness of ϵ in our mathematical consideration. Because of the obvious property $f_n(y) = f_n(-y)$, equation (2.10) implies that the phase differences ϕ_n meet the condition

$$\phi_n(y) = -\phi_n(-y) + 0\mathrm{mod}2\pi. \tag{2.11}$$

The dimensionless Gibbs free energy $\Omega(H)$, normalized via the relation

$$\frac{4\pi\Omega(H)}{H_c^2(T)a\lambda_{J\infty}W_z} \to \Omega(H),$$

in terms of the mean-field quantities f_n, ϕ_n and $H_n(y)$ has the form

$$\Omega(H) = \sum_{n=0}^{N-1}\int_{-L}^{L} dy \left[-f_n^2(y) + \frac{1}{2}f_n^4(y) + \frac{r(T)\epsilon^2}{2}\left(\frac{df_n(y)}{dy}\right)^2\right.$$

$$\left. + \frac{2r(T)}{\epsilon^2 f_n^2(y)}\left[H_{n+1}(y) - H_n(y)\right]^2 + 2r(T)\left[H_n(y) - H\right]^2\right]$$

$$+ \frac{r(T)}{2}\sum_{n=1}^{N-1}\int_{-L}^{L} dy \left[f_{n-1}^2(y) + f_n^2(y) - 2f_n(y)f_{n-1}(y)\cos\phi_n(y)\right]. \tag{2.12}$$

Here, the sum of the three terms in the first line on the right-hand side is the condensation energy, the sum of the two terms in the second line is the electromagnetic energy E_{em}, and the last term is the Josephson energy E_J. The intralayer current in the nth S-layer $J_n(y)$ (normalized to $\bar{H}_s$) and the density of the Josephson current between the nth and the $(n-1)$th S-layers $j_{n,n-1}(y)$ (normalized to j_0) are given by

$$J_n(y) = \frac{1}{4\pi}\left[H_n(y) - H_{n+1}(y)\right], \quad 0 \le n \le N-1, \tag{2.13}$$

and

$$j_{n,n-1}(y) = 2\frac{dH_n(y)}{dy} = f_n(y)f_{n-1}(y)\sin\phi_n(y), \quad 1 \le n \le N-1, \tag{2.14}$$

respectively.

Assuming that the temperature range satisfies the condition of the weak-coupling limit

$$r(T) \ll 1, \tag{2.15}$$

one can obtain a perturbative solution for f_n and ϕ_n up to any desired order in $r(T)$, starting from the zero-order solution to (2.5), (2.6),

$$f_n = 1, \tag{2.16}$$

and the zero-order equations for ϕ_n,

$$H_{n+1}(y) - \left(2+\epsilon^2\right) H_n(y) + H_{n-1}(y) = -\frac{\epsilon^2}{2}\frac{d\phi_n(y)}{dy}, \quad 1 \le n \le N-1, \tag{2.17}$$

where $H_n(y)$ are given by (2.10) with $f_n = 1$ and satisfy the boundary conditions (2.8). For most applications, it is sufficient to consider expressions for physical quantities only in leading order in $r(T)$. Thus, for example, substituting (2.16) and the solution of (2.17) into (2.12) immediately yields a first-order expansion for the Gibbs free energy, because first-order corrections to the condensation-energy term cancel out.

A detailed mathematical analysis of Eqs. (2.17) is the subject of section III. Here we point out that these equations can be transformed into a very useful for application form by solving for $H_n(y)$ (see Appendix A for mathematical details):

$$H_n(y) = h_n(y) + H_n, \tag{2.18}$$

$$h_n(y) = \frac{\epsilon^2}{2}\sum_{m=1}^{N-1} G(n,m)\frac{d\phi_m(y)}{dy}, \tag{2.19}$$

$$H_n = \frac{H\left(\mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n}\right)}{\mu^{-N} - \mu^N}, \tag{2.20}$$

where $G(n,m)$ are given by (A9), and μ is given by (A5). By (A8), and (2.9), (A13), expression (2.18) explicitly satisfies boundary conditions (2.8) and $H_n(\pm L) = H$. Moreover, the y-independent quantities H_n in (2.18) have clear physical meaning: Being solutions of (2.17) with $\frac{d\phi_m}{dy} \equiv 0$, they describe distribution of the local magnetic field within the I-layers in the homogeneous Meissner state. (See section III of Ref.[2].) [Note that in an infinite layered superconductor $H_n \equiv 0$, and $\sum\limits_{m=1}^{N-1} G(n,m)\ldots \to \sum\limits_{m=-\infty}^{+\infty} G_\infty(n,m)\ldots$, where $G_\infty(n,m)$ are defined by (A17).] By comparing Eqs. (2.18) with Eqs. (2.10) (where $f_n = 1$), using the property $H_n = H_{N-n}$ and (A11), we establish the symmetry relations

$$\phi_n(y) = \phi_{N-n}(y), \quad h_n(y) = h_{N-n}(y), \quad H_n(y) = H_{N-n}(y). \tag{2.21}$$

Relations (2.21) replace the relations $\phi_n(y) = \phi_{n+1}(y) \equiv \phi(y)$ and $h_n(y) = h_{n+1}(y) \equiv h(y)$ of the infinite layered superconductor.[2]

By the use of Eqs. (2.18), the Gibbs free energy up to first order in $r(T) \ll 1$ can be given the form

$$\Omega(H) = -\frac{NW}{2} + r(T)\left[\frac{2H^2W}{H_s^2}(N-1)\right.$$

$$+\frac{\epsilon^2}{2}\sum_{n=1}^{N-1}\sum_{m=1}^{N-1} G(n,m)\int\limits_{-L}^{L} dy \frac{d\phi_n(y)}{dy}\frac{d\phi_m(y)}{dy}$$

$$\left. +2\sum_{n=1}^{N-1}\int\limits_{-L}^{L} dy \sin^2\frac{\phi_n(y)}{2} - 4H\sum_{n=1}^{N-1}\Phi_n\frac{\phi_n(L)-\phi_n(-L)}{2\pi}\right], \tag{2.22}$$

$$\Phi_n = \pi\left[1 - \frac{\mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n}}{\mu^{-N} - \mu^N}\right], \tag{2.23}$$

where H_s is the superheating (penetration) field defined in Eq. (A14), and Φ_n is the total flux carried by a Josephson vortex positioned in the nth I-layer. (The quantities H_s and Φ_n are thoroughly discussed in section IIIB.) The

quadratic form of the electromagnetic energy [the second line on the right-hand side of Eq. (2.22)] can be diagonalized with the help of (A15), (A16):

$$E_{em} = \frac{1}{N} \sum_{k=0}^{\left[\frac{N}{2}\right]-1} \lambda_{2k+1}^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} \sin \frac{\pi n\,(2k+1)}{N} \sin \frac{\pi m\,(2k+1)}{N} \int_{-L}^{L} dy \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy}, \tag{2.24}$$

where $[u]$ is the integer part of u. [In deriving (2.24), we employed the symmetry relations (2.21).] Equation (2.24) explicitly shows that physical solutions in a stack with N S-layers are characterized by $\left[\frac{N}{2}\right]$ Josephson length scales $\lambda_{Jk} \equiv \lambda_{2k+1}$ ($k = 0, 1, \ldots, \left[\frac{N}{2}\right] - 1$) from the set (A15).

III. MAJOR RESULTS

A. Analysis of the equations for the phase differences

By differentiation with respect to y, integrodifferential equations (2.17) reduce to a system of $N-1$ ordinary nonlinear second-order differential equations

$$\frac{d^2\phi_1(y)}{dy^2} = \frac{1}{\epsilon^2}\left[\left(2+\epsilon^2\right)\sin\phi_1(y) - \sin\phi_2(y)\right],$$

$$\frac{d^2\phi_n(y)}{dy^2} = \frac{1}{\epsilon^2}\left[\left(2+\epsilon^2\right)\sin\phi_n(y) - \sin\phi_{n+1}(y) - \sin\phi_{n-1}(y)\right], \quad 2 \le n \le N-2,$$

$$\frac{d^2\phi_{N-1}(y)}{dy^2} = \frac{1}{\epsilon^2}\left[\left(2+\epsilon^2\right)\sin\phi_{N-1}(y) - \sin\phi_{N-2}(y)\right] \tag{3.1}$$

with boundary conditions (2.9). As shown in section II, physical solutions to (3.1) necessarily obey the symmetry relations (2.11) and (2.21).

A detailed analysis of the analytical structure of the coupled static SG equations (3.1) is the main task of this subsection. Such differential equations were first derived within the framework of a less rigorous, phenomenological approach in Ref.[3] . Particular numerical solutions (for $H = 0$) were obtained in several publications.[3,5–7] However, any analytical investigation of these equations has not been carried out up until now.

1. Existence and uniqueness of the solution to the initial value problem

Consider Eqs. (3.1) on the whole axis $-\infty < y < +\infty$. Two simple properties of (3.1) are quite obvious: If $\phi_n(y)$ ($1 \le n \le N-1$) is a solution, the functions $\bar{\phi}_n(y)$ given by

$$\bar{\phi}_n(y) = \phi_n(y) + 2\pi k \quad (k \text{ is an integer}), \tag{3.2}$$

and

$$\bar{\phi}_n(y) = \phi_n(y+c) \quad (c \text{ is an arbitrary constant}) \tag{3.3}$$

are also solutions. [The latter is a result of the fact that y does not enter explicitly the right-hand side of (3.1).] Our conclusions about the solution to (3.1) will be substantially based on another key property, which we formulate as a lemma:

Lemma 1. Consider an arbitrary interval $I = [L_1, L_2]$ and $y_0 \in I$. The initial value problem for Eqs. (3.1) with arbitrary initial conditions $\phi_n(y_0) = \alpha_n$, $\frac{d\phi_n}{dy}(y_0) = \beta_n$ has a unique solution in the *whole* interval I. This solution has continuous derivatives with respect to y of arbitrary order and continuously depends on the initial data. (For the proof of **Lemma 1**, see Appendix B.)

It is worth noting that the existence and uniqueness of a smooth solution to the initial value problem in the *whole* interval I is rather nontrivial for *nonlinear* differential equations: For such equations, theorems of existence and

uniqueness are usually valid only locally, in the neighborhood of initial data.[8] In our case, global character of the solution and its infinite differentiability are ensured by the fact that ϕ_n enter the right-hand side of Eqs. (3.1) only as arguments of the sine. Note that because of the arbitrariness of the interval I, the solution can be uniquely continued onto the whole axis $-\infty < y < +\infty$.[8]

Differentiating (2.18) with respect to y yields

$$\sin\phi_n(y) = \epsilon^2 \sum_{m=1}^{N-1} G(n,m) \frac{d^2\phi_m(y)}{dy^2}. \tag{3.4}$$

Multiplying (3.4) by $\frac{d\phi_n(y)}{dy}$, summing over the layer index n with the use of (A10) and performing integration, we arrive at the first integral of Eqs. (3.1):

$$C\,(H) - \sum_{n=1}^{N-1} \cos\phi_n(y) = \frac{\epsilon^2}{2} \sum_{n=1}^{N-1}\sum_{m=1}^{N-1} G(n,m) \frac{d\phi_n(y)}{dy}\frac{d\phi_m(y)}{dy}, \tag{3.5}$$

$$C\,(H) = \frac{2H^2}{H_s^2}\,(N-1) + \sum_{n=1}^{N-1} \cos\phi_n(L), \tag{3.6}$$

where $C\,(H)$ is the constant of integration. Using (3.5), one can eliminate the electromagnetic-energy term from the Gibbs free energy (2.22).

The existence of the first integral of Eqs. (3.1) was first established in Ref.[5]. However, the explicit form of the matrix elements $G(n,m)$ and the actual value of $C\,(H)$ was not determined by the authors of Ref.[5]. Unfortunately, the existence of the first integral for the *infinite* ($N = \infty$) layered superconductor was not noticed in any previous publications, which partly explains the difficulties with the minimization of the Gibbs free energy.[17,18,15,19] [Equation (3.5) holds for $N = \infty$ too, taking account of the substitution $\sum\limits_{m,n=1}^{N-1} G(n,m)\ldots \to \sum\limits_{m,n=-\infty}^{+\infty} G_\infty(n,m)\ldots$, where $G_\infty(n,m)$ are defined by (A17)]. In the case $N = \infty$, the Gibbs free energy $\Omega\,(H)$ should be additionally minimized with respect to the *phases* φ_n, which by the use of the first integral immediately yields the exact solution[1,2] with $\phi_n(y) = \phi_{n+1}(y) \equiv \phi(y)$.

For our further consideration, we will need an auxiliary lemma:

Lemma 2. Consider Eqs. (3.1) on the whole axis $-\infty < y < +\infty$. Conditions

$$\phi_n(y_0) = 0, \qquad \frac{d\phi_n}{dy}(y_0) = 0, \qquad n = 1, 2, \ldots, N-1, \tag{3.7}$$

where $y_0 \neq \pm\infty$, cannot by satisfied simultaneously for all n by any nontrivial solution of (3.1).

The proof of **Lemma 2** is straightforward: Indeed, conditions (3.7) specify the initial value problem for (3.1). This initial value problem has the trivial solution $\phi_1 = \phi_2 = \ldots = \phi_{N-1} \equiv 0$. By **Lemma 1**, this solution is unique and there are no nontrivial solutions that satisfy (3.7).

The importance of **Lemma 2** lies in the fact that it prohibits the existence of any topological (vortex) solutions in any finite interval $[L_1, L_2]$ in the absence of an external field ($H = 0$). Indeed, any such solution must necessarily satisfy the boundary conditions[22] $\phi_n(L_1) = 0$ and $\phi_n(L_2) = 0 \mathrm{mod} 2\pi$ for all n, with $\phi_n(L_2) \neq 0$ for at least one $n = m$. For $H = 0$, we also have $\frac{d\phi_n}{dy}(L_1) = \frac{d\phi_n}{dy}(L_2) = 0$ for all n. However, the former and the latter conditions are incompatible by virtue of **Lemma 2**.

Another important consequence of **Lemma 2** is that for any finite interval $[-L, L]$ the constant of integration in (3.5) satisfies the inequality

$$C(H) > N - 1, \tag{3.8}$$

for any $H > 0$. Indeed, for the Meissner solution we have $\phi_n(y) = -\phi_n(-y)$ and $\phi_n(0) = 0$ [see (2.11)]. By setting $y = 0$ in (3.5), applying **Lemma 2** and using the fact that the quadratic form on the right-hand side of (3.5) is positively definite, we get (3.8). For topological solutions, inequality (3.8) is quite obvious.

Alongside solutions to (3.1) in a finite interval $[-L, L]$, we will consider physical solutions to (3.1) in the infinite ($-\infty < y < +\infty$) and a semiinfinite ($0 \leq y < +\infty$) intervals. To ensure the convergence of the integrals over y in the expression for the free energy [see (2.22)], such solutions must necessarily satisfy the asymptotic boundary conditions[22,23]

$$\phi_n(y) \underset{y \longrightarrow \pm\infty}{\longrightarrow} 0 \bmod 2\pi, \tag{3.9}$$

$$\frac{d\phi_n(y)}{dy} \underset{y \longrightarrow \pm\infty}{\longrightarrow} 0, \tag{3.10}$$

for any $1 \leq n \leq N-1$. (Note that **Lemma 2** does not obtain for $y_0 = \pm\infty$.) These solutions with a square-integrable derivative will be called *localized*. Their consideration can be reduced to the consideration of the standard initial value problem at $y = 0$, with the initial data subject to a certain existence condition.

By inserting (3.9) and (3.10) into (3.5), we find the value of the constant of integration: $C\,(0) = N - 1$. [Compare with inequality (3.8) for $L < \infty$, $H > 0$.] Thus Eq. (3.5) becomes

$$\sum_{n=1}^{N-1} \sin^2 \frac{\phi_n(y)}{2} = \frac{\epsilon^2}{4} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n,m) \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy}. \tag{3.11}$$

Substituting initial values $\phi_n(0) = \alpha_n$, $\frac{d\phi_n}{dy}(0) = \beta_n$ into (3.11), we obtain the desired condition on α_n and β_n that we call the *criterion of the existence* of localized solutions:

$$\sum_{n=1}^{N-1} \sin^2 \frac{\alpha_n}{2} = \frac{\epsilon^2}{4} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n,m) \beta_n \beta_m. \tag{3.12}$$

Indeed, **Lemma 1** guarantees the existence, uniqueness and differentiability of a solution for arbitrary α_n and β_n in the whole interval $(-\infty, +\infty)$ [or $[0, +\infty)$]. Owing to our special choice of the constant of integration in (3.11), the solution determined by α_n and β_n obeying (3.12) will necessarily satisfy asymptotic conditions (3.9), (3.10). Moreover, this solution will automatically satisfy the conditions

$$\frac{d^k\phi_n(y)}{dy^k} \underset{y \to \pm\infty}{\to} 0, \tag{3.13}$$

for any $1 \leq n \leq N-1$ and $2 \leq k$, which can be verified by repeated differentiation of (3.4) and application of (3.9), (3.10).

Note that localized solutions are characterized by the absence of any electromagnetic effects at $y \to \pm\infty$. Furthermore, owing to (3.11), the Josephson energy of localized solutions E_J is exactly equal to their electromagnetic energy E_{em}. (From a field-theoretical point of view, this fact is a manifestation of the virial theorem.[23])

3. *Solutions with periodic derivatives. The major* **Theorem**

Our conclusions about the type and the properties of physical solutions for $H > 0$ will be based on the major **Theorem**:

Theorem. Consider Eqs. (3.1) on the whole axis $-\infty < y < +\infty$. The initial value problem

$$\phi_n(y_0) = 0, \qquad \frac{d\phi_n}{dy}(y_0) = 2H > 0, \qquad n = 1, 2, \ldots, N-1, \tag{3.14}$$

for (3.1) has a unique solution on the whole axis $-\infty < y < +\infty$. This solution is characterized by the properties

$$\phi_n\,(y + P) = \phi_n\,(y) + 2\pi, \tag{3.15}$$

$$\frac{d\phi_n}{dy}\,(y + P) = \phi_n\,(y) > 0, \tag{3.16}$$

$$\phi_n\left(y_0+\frac{1}{2}P\right)=\pi, \tag{3.17}$$

$$\frac{d\phi_n}{dy}\left(y_0+\frac{1}{2}P\right)=2\sqrt{H^2+H_s^2}, \qquad \text{for all } n=1,2,\ldots,N-1, \tag{3.18}$$

where H_s is defined via (A14), and the period P is given by

$$P(H)=\frac{2}{\sqrt{H^2+H_s^2}}\int\limits_{(0,0,\ldots,0)}^{\left(\frac{\pi}{2},\frac{\pi}{2},\ldots,\frac{\pi}{2}\right)}\frac{ds}{\sqrt{1-\frac{1}{N-1}\frac{H_s^2}{H^2+H_s^2}\sum\limits_{n=1}^{N-1}\sin^2\theta_n}}, \tag{3.19}$$

$$ds=\epsilon H_s\sqrt{\frac{1}{N-1}\sum_{n=1}^{N-1}\sum_{m=1}^{N-1}G(n,m)d\theta_n d\theta_m}.$$

The integration in (3.19) is performed along the integral curve:

$$\theta_n=\frac{\phi_n(y)}{2}, \qquad y_0\le y\le y_0+\frac{1}{2}P, \qquad n=1,2,\ldots,N-1,$$

with ds being the length increment. (See Appendix C for the proof.)

An immediate physical consequence of the **Theorem** is the absence of single-vortex solutions to (3.1) in a finite interval $[-L,L]$ for any $H>0$. (For $H=0$, the absence of single-vortex solutions in a finite interval follows from **Lemma 2**.) Indeed, if such solutions existed, they would satisfy at a certain $H=H_1>0$ the boundary conditions[22]

$$\phi_n(-L)=0, \qquad \frac{d\phi_n}{dy}(-L)=2H_1>0, \qquad n=1,2,\ldots,N-1,$$

$$\phi_l(L)=2\pi, \quad \frac{d\phi_l}{dy}(L)=2H_1>0; \quad \phi_n(L)=0, \quad \frac{d\phi_n}{dy}(L)=2H_1>0, \quad n\ne l.$$

However, these boundary conditions are mutually incompatible: The boundary conditions at $y=-L$ specify the initial value problem for (3.1). By the **Theorem**, this initial value problem admits a unique solution with $\frac{d\phi_n}{dy}>0$ on the *whole* axis $-\infty<y<+\infty$ for *all* n, whereas the boundary conditions at $y=L$ imply that $\frac{d\phi_n}{dy}$ with $n\ne l$ change the sign twice in the interval $[-L,L]$.

Analogously, one can prove the absence for $H>0$ and $L<\infty$ of any other incoherent vortex solutions, as well as such vortex configurations as the "triangular lattice"[14–16] and the "row mode".[6] In contrast, the properties (3.15)-(3.19) are inherent to the vortex-plane solutions.[1,2]

Note that $P(H)\underset{H\to 0}{\longrightarrow}\infty$, in agreement with **Lemma 2**. On the other hand, for $H\gg H_s$, the path of integration in (3.19) is a straight line: $\theta_n=\frac{\pi}{P}(y-y_0)$, $y_0\le y\le y_0+\frac{1}{2}P$, $n=1,2,\ldots,N-1$, which yields $P(H)\underset{H\gg H_s}{\longrightarrow}\frac{\pi}{H}$.

For a single thin-layer junction ($N=2$), equations (3.1) reduce to a single SG equation for $\phi_1\equiv\phi$, with $\lambda_{J0}=H_s^{-1}\equiv\frac{\epsilon}{\sqrt{2+\epsilon^2}}$. In agreement with Refs.[24,25] , the period (3.19) becomes

$$P(H)=\frac{2}{\sqrt{H^2+H_s^2}}K\left(\frac{H_s^2}{H^2+H_s^2}\right), \tag{3.20}$$

where $K\left(k^2\right)$ is the complete elliptic integral of the first kind.[26] Relations (3.15)-(3.19), applied to ϕ, determine an *exact* vortex solution in terms of the Jakobi elliptic functions $\mathrm{am}(u)$ and $\mathrm{dn}(u)=\frac{d}{du}\mathrm{am}(u)$.[26]

The true vortex-plane solutions appear in a double-junction stack ($N=3$). Owing to the symmetry relations (2.21), equations (3.1) again reduce to a single SG equation for $\phi_1=\phi_2\equiv\phi$, with $\lambda_{J0}=H_s^{-1}\equiv\frac{\epsilon}{\sqrt{1+\epsilon^2}}$. The period P is again given by (3.20) (with redefined H_s), and the *exact* vortex-plane solution is again expressed via the functions $\mathrm{am}(u)$ and $\mathrm{dn}(u)$. These two exactly-solvable examples ($N=2$ and $N=3$) not only demonstrate the power and generality of the **Theorem**, but also show that the vortex-plane solutions are a natural generalization of the well-known vortex solutions in a single junction.[24,25,27] (See section IV for a more detailed discussion of the cases $N=2,3$.)

B. Meissner solutions. The superheating (penetration) fields H_{sL} and $H_s \equiv H_{s\infty}$

In a finite interval $[-L, L]$, the Meissner solution is characterized by the relations

$$\phi_n(y) = -\phi_n(-y), \qquad 1 \le n \le N-1, \tag{3.21}$$

resulting from the general symmetry (2.11), and obeys the conditions (2.21). Thus, the Meissner boundary value problem is completely specified by the boundary conditions

$$\frac{d\phi_n}{dy}(-L) = 2H > 0, \qquad \phi_n(0) = 0, \qquad 1 \le n \le N-1. \tag{3.22}$$

Up to a certain field $H = H_{sL}$, the boundary value problem (3.22) has a unique solution with $\frac{d\phi_n}{dy} > 0$ in the whole interval $[-L, L]$ and $-\pi \le \phi_n(-L) < 0$ for all n. The field H_{sL} is determined from the conditions $\phi_n(\pm L) = \pm\pi$ for all n. [As follows from (3.6), (3.8), $H_{sL} > H_s$.] Indeed, for $H \ll H_s$ the Meissner solution can be obtained in a closed analytical form as a linear combination of $\left[\frac{N}{2}\right]$ exponentials with λ_{Jk} [see (2.24)]: According to (3.6), (3.8), this case allows of linearization. The existence of a unique Meissner solution for higher values of H follows from continuous dependence of the solution on the initial data $\phi_n(-L) = \alpha_n$ and $\frac{d\phi_n}{dy}(-L) = 2H \equiv \beta_n$. (See **Lemma 1**.) The Meissner solution ceases to exist at $H = H_{sL}$: For this field, both $\frac{d\phi_n}{dy}(y)$ and $h_n(y)$ are local maxima at $y = \pm L$. [See (2.19), (3.4).]

Physically, the conditions $\phi_n(\pm L) = \pm\pi$ correspond to the vanishing of the surface barrier induced by Josephson currents $[j_{n,n-1}(\pm L) = 0]$ and the formation of a vortex plane at the side boundaries of the stack.[1] (Thus, at $H = H_{sL}$, the total flux due to the field penetration through the $y = \pm L$ interfaces is exactly equal to the flux carried by a single vortex plane.) For $H > H_{sL}$, only topological vortex-plane solutions are possible. However, the vortex-plane solutions can exist and be favorable energetically already at fields $H < H_{sL}$. [See the next subsection.] For these reasons, H_{sL} should be identified both with the *superheating* field of the Meissner state and the *penetration* field for the vortex planes. According to the **Theorem**, at $H = H_{sL}$, $\frac{d\phi_n}{dy}(0) = 2\sqrt{H_{sL}^2 - H_s^2}$ for all n, and hence $H_n(0) = \sqrt{H_{sL}^2 - H_s^2} + \left(H_{sL} - \sqrt{H_{sL}^2 - H_s^2}\right)[G(n,1) + G(n, N-1)]$. The field H_{sL} itself is determined by the implicit equation

$$2L = P\left(\sqrt{H_{sL}^2 - H_s^2}\right), \tag{3.23}$$

where P is given by (3.19), with the path of integration being determined by the solution to the boundary value problem $\phi_n(-L) = -\pi$, $\phi_n(0) = 0$. Note that $H_{sL} \to H_s$ for $L \gg H_s^{-1}$, which will be explained below.

The consideration of the Meissner effect becomes simpler, when $L \gg \lambda_{J\,\max} \equiv \lambda_{J0}$. In this case, the interval $[-L, L]$ can be transformed into $[0, +\infty)$ by changing the variable $y \to y - L$ and proceeding to the limit $L \to \infty$. In the semiinfinite interval $[0, +\infty)$, the Meissner solution necessarily satisfies the conditions $\phi_n(y) \underset{y\to+\infty}{\longrightarrow} 0$, $\frac{d\phi_n(y)}{dy} \underset{y\to+\infty}{\longrightarrow} 0$, and we can use the criterion (3.12) that now takes the form

$$\frac{1}{N-1}\sum_{n=1}^{N-1} \sin^2\frac{\phi_n(0)}{2} = \frac{H^2}{H_s^2}, \tag{3.24}$$

where, by (A14) and (A15), (A16),

$$H_s = \left[1 - \frac{\left(2\sqrt{1+\frac{\epsilon^2}{4}} - \epsilon\right)\left(1 - \mu^{N-1}\right)}{\epsilon\,(N-1)\left(1 + \mu^{N-1}\right)}\right]^{-\frac{1}{2}}$$

$$\equiv \sqrt{\frac{(N-1)\,N}{2}}\left[\sum_{k=0}^{\left[\frac{N}{2}\right]-1} \lambda_{Jk}^2 \cot^2\frac{\pi\,(2k+1)}{2N}\right]^{-\frac{1}{2}}. \tag{3.25}$$

Physical interpretation of the quantity H_s is straightforward. The maximum value of the left-hand side of (3.24) is achieved when all $\phi_n(0) = -\pi$ and is equal to unity, which corresponds to $H = H_s$ on the right-hand side. Hence H_s should be identified with the superheating (penetration) field $H_{s\infty}$ ($H_s \equiv H_{s\infty}$). Note that H_s for $N < \infty$ is always higher than the corresponding field[1,2] $\bar{H}_s = 1$ of the infinite ($N = \infty$) layered superconductor. For $N \to \infty$, $H_s \to 1$.

According to (3.24), equations (3.1) can be linearized for $H \ll H_s$. In this limit, the explicit Meissner solution in the interval $[0, +\infty)$ is

$$\phi_n(y) = -\frac{4H}{N} \sum_{k=0}^{\left[\frac{N}{2}\right]-1} \lambda_{Jk} \cot\frac{(2k+1)\pi}{2N} \sin\frac{(2k+1)n\pi}{N} \exp\left[-\frac{y}{\lambda_{Jk}}\right], \tag{3.26}$$

$$H_n(y) = \frac{2H}{N} \sum_{k=0}^{\left[\frac{N}{2}\right]-1} \lambda_{Jk}^2 \cot\frac{(2k+1)\pi}{2N} \sin\frac{(2k+1)n\pi}{N} \exp\left[-\frac{y}{\lambda_{Jk}}\right] + H_n, \tag{3.27}$$

where H_n is given by (2.20). The distribution of the currents J_n and $j_{n,n-1}$ can be easily obtained from (2.13), (2.14) using (3.26), (3.27).

It is instructive to investigate a transition to the infinite layered-superconductor limit $N - 1 \gg 2\left[\epsilon^{-1}\right]$ ($\epsilon < 1$) in Eqs. (3.26), (3.27). Thus, for I-layers whose layer index n satisfies the condition

$$\left[\epsilon^{-1}\right] \ll n \ll N - 1 - \left[\epsilon^{-1}\right], \tag{3.28}$$

we have $H_n = H\left(\mu^n + \mu^{N-n}\right) \ll H$. Under the condition (3.28), the main contribution to the sums over k in (3.26), (3.27) comes from $k \ll \left[\frac{N}{2}\right] - 1$. For such k and $N \gg 1$, $\lambda_{Jk} \approx \lambda_{J0} \approx \bar{\lambda}_J = 1$. Therefore, for n in the interval (3.28) we get

$$\phi_n(y) \approx -\frac{8H}{\pi} \exp(-y) \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} = -2H \exp(-y),$$

$$H_n(y) \approx H \exp(-y),$$

in complete agreement with the results of Refs.[1,2] .

Two final remarks would be in order here. The fact that the Meissner solution in a stack with N S-layers is characterized by $\left[\frac{N}{2}\right]$ Josephson lengths λ_{Jk} was not noticed in any previous publications. Moreover, in contrast to continuum type-II superconductors,[28] the local field $H_n(y)$ at $H = H_s$ cannot be represented as a sum of a purely Meissner field and a vortex field, because the principle of superposition does not hold for the nonlinear Eqs. (3.1). Unfortunately, the latter point was not realized in Ref.[29] concerned with infinite layered superconductors.

C. Vortex-plane solutions. The lower critical field H_{c1}

As explained in the Introduction, by a vortex plane we understand a coherent chain of $N - 1$ Josephson vortices (one vortex per each I-layer) positioned in a plane parallel to the coordinate axes x, z: see Fig. 2. (In this plane, all $\frac{d\phi_n}{dy}$ and h_n are local maxima.) Such solutions in an interval $[-L, L]$ obey the general relations (2.11) with the *same* constant $0 \bmod 2\pi \neq 0$ for *all* n and the symmetry conditions (2.21): hence the same set of $\left[\frac{N}{2}\right]$ Josephson lengths λ_{Jk} as in the Meissner case. The **Theorem** envisages the existence of vortex-plane solutions for any $H > 0$. Using the **Theorem** and **Lemma 1**, we can easily establish all basic properties of the vortex-plane solutions. They can be summarized as follows.

The boundary value problem for the vortex-plane solutions is specified by the boundary conditions

$$\frac{d\phi_n}{dy}(-L) = 2H > 0, \qquad \phi_n(0) = \pi N_v, \qquad 1 \le n \le N-1, \qquad N_v = 1, 2, \ldots, \tag{3.29}$$

where N_v is the number of vortex planes. As in the case of the Meissner boundary value problem (3.22), the boundary value problem (3.29) for a fixed N_v and any $L > 0$ has a unique solution, with $\frac{d\phi_n}{dy} > 0$ in the whole interval $[-L, L]$ and $-\pi \le \phi_n(-L) \le 0$ for all n, in a certain field range

$$\sqrt{H_{N_v-1}^2 - H_s^2} \le H \le H_{N_v}, \tag{3.30}$$

where $H_0 \equiv H_{sL}$. The lower bound of the existence of the N_v-vortex-plane solution is determined by the boundary value problem

$$\phi_n(-L) = 0, \qquad \phi_n(0) = \pi N_v, \qquad 1 \le n \le N-1, \tag{3.31}$$

whereas the upper bound is determined by the boundary value problem

$$\phi_n(-L) = -\pi, \qquad \phi_n(0) = \pi N_v, \qquad 1 \le n \le N-1. \tag{3.32}$$

The field H_{N_v} is determined by the implicit equation

$$2L = (N_v + 1)\, P\left(\sqrt{H_{N_v}^2 - H_s^2}\right), \tag{3.33}$$

where P is given by (3.19), with the path of integration being determined by the solution to the boundary value problem (3.32). Note that for $N_v = 0$ (the Meissner state) Eq. (3.33) coincides with Eq. (3.23), as it should. Physically, P determines the period of the distribution of $H_n(y)$ and $j_{n,n-1}(y)$.

As follows from (3.30), the range of the existence of the state with N_v vortex planes overlaps with that of the state with $N_v - 1$ vortex planes. Thus, the state with $N_v = 1$ overlaps with the Meissner state. For $L \gg H_s^{-1}$, the overlapping can occur for several neighboring states. The overlapping is negligibly small only for $L \ll H_s^{-1}$ (with arbitrary N_v) and for $N_v \gg 1$ (with arbitrary L). (For ordinary Josephson vortices in a single junction, the overlapping is well known.[24,25]) However, the actual number of vortex planes, N_v, for given H should be determined from the requirement that the free energy be an absolute minimum. Note that for $L \to \infty$, $\sqrt{H_0^2 - H_s^2} \equiv \sqrt{H_{sL}^2 - H_s^2} \to 0$ and the state with $N_v = 1$ appears already in zero external field. This case is discussed in more detail below.

For $L \gg H_s^{-1}$, we can proceed to the limit $L \to \infty$ and consider the state with $N_v = 1$ in the interval $(-\infty, +\infty)$. With the asymptotic boundary conditions

$$\phi_n(y) \underset{y\to-\infty}{\to} 0, \quad \phi_n(y) \underset{y\to+\infty}{\to} 2\pi, \qquad \frac{d\phi_n(y)}{dy} \underset{y\to\pm\infty}{\to} 0 \tag{3.34}$$

for all n, this state satisfies the criterion (3.12) and can be obtained as the solution to an initial value problem at $y = 0$. Indeed, it can be constructed from the Meissner solution in the interval $[0, +\infty)$ at $H = H_s$: Using the property (3.2), by means of the transformation $\phi_n \to \phi_n + 2\pi$ we obtain in the interval $[0, +\infty)$ a solution that satisfies the initial conditions

$$\alpha_n \equiv \phi_n(0) = \pi, \quad \beta_n \equiv \frac{d\phi_n}{dy}(0) = 2H_s, \quad 1 \le n \le N-1, \tag{3.35}$$

and the conditions (3.34) for $y \to +\infty$. By **Lemma 1**, this solutions can be uniquely continued into the whole interval $(-\infty, +\infty)$. The solution thus obtained satisfies the conditions (3.34) for $y \to \pm\infty$ and hence represents the desired singe-vortex-plane solution. By the construction, $h_n(0) = H_s\,[1 - G(n,1) - G(n, N-1)]$. The total flux carried by the vortex plane is

$$\Phi = \sum_{n=1}^{N-1} \Phi_n = \pi\,(N-1)\left[1 - \frac{2\sqrt{1+\frac{\epsilon^2}{4}} - \epsilon}{\epsilon\,(N-1)}\,\frac{1-\mu^{N-1}}{1+\mu^N}\right], \tag{3.36}$$

where Φ_n is given by (2.23). Note that in contrast to Josephson junctions with thick electrodes[27] and infinite layered superconductors,[1,2] the flux carried by a Josephson vortex in a finite thin-layer S/I structure *is not quantized* and is always smaller than the flux quantum $\Phi_0 = \pi$. For $N - 1 \gg 2\left[\epsilon^{-1}\right]$, $\Phi \to \pi\,(N-1)$, as it should.

To determine the thermodynamic *lower critical* field H_{c1} at which the vortex-plane solutions for $L \gg H_s^{-1}$ become energetically favorable, we must calculate the difference between the Gibbs free energy in the presence of a single vortex plane, $\Omega_v(H)$, and the Gibbs free energy of the homogeneous Meissner state, $\Omega_M(H)$ [the sum of the phase-independent terms in (2.22)]:

$$\Omega_v(H) - \Omega_M(H)$$

$$= r(T)\left[\epsilon^2\sum_{n=1}^{N-1}\sum_{m=1}^{N-1}G(n,m)\int\limits_{-\infty}^{+\infty}dy\frac{d\phi_n(y)}{dy}\frac{d\phi_m(y)}{dy} - 4\Phi H\right], \tag{3.37}$$

where the total flux Φ is given by (3.36). The first term on the right-hand side of (3.37) should be interpreted as the self-energy of the vortex plane: $E_v = 2E_{em} = 2E_J$. From (3.37), we get:

$$H_{c1} = \frac{E_v}{4r(T)\Phi}. \tag{3.38}$$

For thermodynamically stable solutions, we must necessarily have $H_{c1} < H_s$. It is straightforward to verify that this condition is met by the vortex-plane solutions. Using (3.4), (3.34), the initial values $\beta_n = 2H_s$ and integrating by parts, we convert E_v into the form

$$E_v = 2r(T)\left[\sum_{n=1}^{N-1}\int\limits_{+\infty}^{0}dy\phi_n(y)\sin\phi_n(y) - 2H_s\Phi\right]. \tag{3.39}$$

The first term on the right-hand side of (3.39) is positive, because in the region $0 \le y < +\infty$ all ϕ_n satisfy the relation $\pi \le \phi_n < 2\pi$. By the use of (3.4) and (A14), we obtain the following strict inequalities:

$$2H_s\Phi < \sum_{n=1}^{N-1}\int\limits_{+\infty}^{0}dy\phi_n(y)\sin\phi_n(y) < 4H_s\Phi,$$

$$0 < E_v < 4r(T)H_s\Phi.$$

Hence,

$$0 < H_{c1} < H_s,$$

as anticipated. Note that in all special cases admitting exact analytical solutions ($N = \infty$,[1,2] and $N = 2, 3$, see section IV), $H_{c1} = \frac{2}{\pi}H_s$.

D. Single-vortex solutions for $H = 0$, $L = \infty$, and other localized incoherent vortex solutions

As shown in section IIIA, the only topological (vortex) solutions admitted by Eqs. (3.1) for $H > 0$ are the vortex-plane solutions. For $H = 0$, equations (3.1) do not possess any topological solutions in a finite interval $[-L, L]$. Therefore, incoherent vortex solutions to (3.1) can exist only for $H = 0$, in the infinite interval $(-\infty, +\infty)$, and must necessarily meet the requirements imposed on localized solutions. Below we consider in detail the most important type of such solutions, namely single-vortex solutions.

A single Josephson vortex positioned in the lth I-layer at $y = 0$ obeys the symmetry relations

$$\phi_l(y) = 2\pi - \phi_l(-y); \quad \phi_n(y) = -\phi_n(-y), \quad n \neq l, \tag{3.40}$$

[see (2.11)] and the asymptotic boundary conditions

$$\phi_l(y) \underset{y\to-\infty}{\to} 0, \quad \phi_l(y) \underset{y\to+\infty}{\to} 2\pi; \qquad \phi_n(y) \underset{y\to\pm\infty}{\to} 0, \quad n \neq l, \tag{3.41}$$

$$\frac{d\phi_n(y)}{dy} \underset{y\to\pm\infty}{\to} 0, \quad \text{for all } 1 \le n \le N-1. \tag{3.42}$$

Moreover, $\frac{dh_n(y)}{dy} > 0$ in the region $-\infty < y < 0$, and $\frac{dh_n(y)}{dy} < 0$ in the region $0 < y < +\infty$. Hence, ϕ_n must satisfy the relations

$$0 < \phi_n(y) < \pi,\ y \in (-\infty, 0)\,; \quad -\pi < \phi_n(y) < 0,\ y \in (0, +\infty)\,, \quad \text{for } n \neq l, \tag{3.43}$$

and the initial conditions

$$\alpha_l \equiv \phi_l(0) = \pi; \quad \alpha_n \equiv \phi_n(0) = 0, \quad n \neq l, \tag{3.44}$$

$$\beta_l \equiv \frac{d\phi_l(0)}{dy} > 0; \quad \beta_n \equiv \frac{d\phi_n(0)}{dy} < 0, \quad n \neq l. \tag{3.45}$$

A necessary condition of the existence of such a solution is provided by the general criterion (3.12) and has the form

$$\frac{\epsilon^2}{4}\sum_{n=1}^{N-1}\sum_{m=1}^{N-1} G(n,m)\beta_n\beta_m = 1. \tag{3.46}$$

The flux through the nth I-layer due to the vortex in the lth I-layer can be found using (2.19) and (3.41):

$$\Phi_{nl} = \int_{-\infty}^{+\infty} dy h_n(y) = \frac{\pi\epsilon}{2\sqrt{1+\frac{\epsilon^2}{4}}}\left[\mu^{|n-l|} - \frac{\mu^n\left(\mu^{l-N}-\mu^{N-l}\right)+\mu^{N-n}\left(\mu^{-l}-\mu^{l}\right)}{\mu^{-N}-\mu^{N}}\right]. \tag{3.47}$$

The total flux carried by the vortex in the lth layer is $\Phi_l = \sum_{n=1}^{N-1}\Phi_{nl}$, where Φ_l is given by (2.23) with $n = l$. For $N < \infty$, the total flux Φ_l is not quantized and is less than the flux quantum $\Phi_0 = \pi$. (See the previous subsection.)

In contrast to the Meissner and the vortex-plane solutions, single-vortex solutions, in general, do not obey the symmetry (2.21), inherent to the original integrodifferential equations (2.7), (2.17) minimizing the Gibbs free-energy. Therefore, they are characterized by the full set (A15) of $N-1$ length scales λ_i. (The only exclusion is a stack with an odd number of junctions $N-1$ and $l = \frac{N}{2}$.)

Although the energy of single-vortex solutions is lower than the self-energy of the vortex-plane solutions at $H = 0$, the former solutions cannot be used, e.g., for estimates of H_{c1} because of their absolute instability at $H > 0$. Unfortunately, this crucial issue was not understood in any previous publications. Thus, the main distinctive feature of single-vortex solutions that precludes their existence at $H > 0$, i.e., $\frac{d\phi_n(0)}{dy} < 0$ for $n \neq l$, was overlooked in Refs.[17–19] concerned with infinite layered superconductors. (The approach of Refs.[17–19] was criticized from different points of view in Refs.[20,1,2] .) The negative sign of $\frac{d\phi_n(0)}{dy}$ for $n \neq l$ was noticed in Ref.[15] . However, the effect of this property on the stability of single-vortex solutions at $H > 0$ was not realized therein.

The feature (3.45) is clearly reproduced by the numerical studies[3,6,7] of finite stacks. Unfortunately, in the absence of any analytical investigations of Eqs. (3.1), the authors of Refs.[3,6,7] failed to establish the actual domain of validity of their numerical results: $H = 0$, $L = \infty$. The condition (3.46) was not found either. In this regard, it would be appropriate to point out an intrinsic limitation of the numerical calculations that does not allow one to consider them as a proof of the existence of single-vortex solutions even in the case $H = 0$. By necessity, these calculations are performed in a *finite* interval $[L_1, L_2]$. However, as shown in section IIIA, equations (3.1) do not admit *any* topological solutions (in a strict mathematical sense[22,23]) in any finite interval at $H = 0$. It should be also noted that the single-vortex solution cannot be represented by a linear combination of the terms $4\arctan\exp\left[\frac{y}{\lambda_i}\right]$,[6,7] because the principle of superposition does not apply to the nonlinear Eqs. (3.1).

The consideration of other localized incoherent vortex solutions to (3.1) (i.e., solutions with $2 \leq k < N-1$ vortices in the plane $y = 0$, and vortex-antivortex configurations) can be done along the same lines. At $H > 0$, all these solutions are unstable too.

IV. THE TWO EXACTLY-SOLVABLE CASES

Below, we present an exact solution for the cases $N = 2, 3$. Valid for any $H \geq 0$ and $L > 0$, this solution provides a clear illustration of the general results of sections IIIA-IIIC. The single-junction case ($N = 2$) illuminates some typical specific features of the thin-S-layer limit and establishes a profound physical and mathematical relationship between the ordinary Josephson vortices and the vortex-plane solutions. In the double-junction case ($N = 3$), our exact solution describes the true vortex planes, which is of considerable experimental interest: For a double-junction stack, we already have direct observations of Josephson-vortex configurations that can be unambiguously identified with the vortex-plane solutions.[12]

A. A single thin-layer junction ($N = 2$)

In this simplest case, a single phase difference $\phi_1(y) \equiv \phi(y)$ satisfies the usual static SG equation

$$\frac{d^2\phi(y)}{dy^2} = \frac{1}{\lambda_{J0}^2} \sin\phi(y), \tag{4.1}$$

with the Josephson length[24]

$$\lambda_{J0} = \frac{\epsilon}{\sqrt{2+\epsilon^2}}. \tag{4.2}$$

Note that λ_{J0}, given by (4.2), for $\epsilon \ll 1$ is much smaller than the Josephson length of a single junction with thick electrodes, which in our dimensionless units is $\lambda_J = \sqrt{\frac{p}{2\lambda}}$.[24] From (2.20) and (3.25), we get the local field in the homogeneous Meissner state

$$H_1 = \frac{2H}{2+\epsilon^2}, \tag{4.3}$$

and the superheating (penetration) field

$$H_s = \lambda_{J0}^{-1} = \frac{\sqrt{2+\epsilon^2}}{\epsilon}, \tag{4.4}$$

respectively. For $\epsilon \ll 1$, the superheating (penetration) field H_s, given by (4.4), is much higher than the corresponding field[24,27] of a single junction with thick electrodes $H_s = \lambda_J$.

Using the **Theorem**, we immediately find an exact solution to the boundary value problems (3.22) and (3.29):

$$\phi(y) = \pi\left(N_v - 1\right) + 2\mathrm{am}\left(\frac{y}{k\lambda_{J0}} + K\left(k^2\right), k\right), \tag{4.5}$$

$$\mathrm{dn}\left(\frac{L}{k\lambda_{J0}}, k\right) = \frac{\sqrt{1-k^2}}{k\lambda_{J0}H}, \qquad N_v = 2m, \quad m = 0, 1, \ldots; \tag{4.6}$$

$$\phi(y) = \pi N_v + 2\mathrm{am}\left(\frac{y}{k\lambda_{J0}}, k\right), \tag{4.7}$$

$$\mathrm{dn}\left(\frac{L}{k\lambda_{J0}}, k\right) = k\lambda_{J0}H, \qquad N_v = 2m+1, \quad m = 0, 1, \ldots, \tag{4.8}$$

where $\mathrm{am}(u)$ and $\mathrm{dn}(u) = \frac{d}{du}\mathrm{am}(u)$ are the Jakobi elliptic functions, and $K\left(k^2\right)$ is the elliptic integral of the first kind.[26] The range of the existence of the solution with $N_v = 0, 1, \ldots$ Josephson vortices is given by (3.30), where H_{N_v} is determined by the implicit equation

$$\frac{L}{\lambda_{J0}} = \frac{(N_v+1)}{H_{N_v}\lambda_{J0}} K\left(\frac{1}{H_{N_v}^2 \lambda_{J0}^2}\right). \tag{4.9}$$

Note that although the Meissner and the vortex solutions to (4.1) were studied a long time ago,[24,25] a closed-form analytical solution of the type (4.5)-(4.9), valid for any $0 < L$ in the whole field range $0 \leq H \ll H_{c2}$, has not been found up until now. (Apparently for this reason, there exists an erroneous belief[30,7] that Josephson vortices "do not form" in single junctions with small L.) Using asymptotics of $\mathrm{am}(u)$, $\mathrm{dn}(u)$ and $K\left(k^2\right)$,[26] we can obtain from (4.5)-(4.9) all physically interesting limiting cases.

1. *The Meissner solution for* $[0 \leq y < +\infty)$

For the fields $0 \leq H \leq H_s = \frac{\sqrt{2+\epsilon^2}}{\epsilon}$, the Meissner solution in the semiinfinite interval $[0, +\infty)$ can be obtained from (4.5), (4.6) with $N_v = 0$ by changing the variable $y \to y - L$ and proceeding to the limit $L \to \infty$. Using the formulas of section II, up to first order in $r(T) \ll 1$, we have

$$\phi(y) = -4 \arctan \frac{H \exp\left[-\frac{y}{\lambda_{J0}}\right]}{H_s + \sqrt{H_s^2 - H^2}}, \tag{4.10}$$

$$H(y) \equiv H_1(y) = h(y) + H_1, \tag{4.11}$$

$$h(y) \equiv h_1(y) = \frac{2\lambda_{J0} H \left[H_s + \sqrt{H_s^2 - H^2}\right] \exp\left[-\frac{y}{\lambda_{J0}}\right]}{\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 + H^2 \exp\left[-\frac{2y}{\lambda_{J0}}\right]}, \tag{4.12}$$

$$j(y) \equiv j_{1,0}(y) = -4H \left[H_s + \sqrt{H_s^2 - H^2}\right]$$

$$\times \frac{\left[\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 - H^2 \exp\left[-\frac{2y}{\lambda_{J0}}\right]\right] \exp\left[-\frac{y}{\lambda_{J0}}\right]}{\left[\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 + H^2 \exp\left[-\frac{2y}{\lambda_{J0}}\right]\right]^2}, \tag{4.13}$$

$$J(y) \equiv J_0(y) = J_1(y) = \frac{1}{4\pi} \left[H - H_1 - h(y)\right], \tag{4.14}$$

$$f(y) \equiv f_0(y) = f_1(y) = 1 - \frac{r(T)}{2} \left[\lambda_{J0}^{-2} h(y) + \frac{2}{\epsilon^2} \left[h(y) + H_1 - H\right]^2\right]. \tag{4.15}$$

2. *The single-vortex solution for* $(-\infty < y < +\infty)$

The single-vortex solution in the infinite interval $(-\infty, +\infty)$ can be obtained from (4.7), (4.8) with $N_v = 1$ by proceeding to the limit $L \to \infty$. It has the form

$$\phi(y) = 4 \arctan \exp\left[\frac{y}{\lambda_{J0}}\right], \tag{4.16}$$

$$h(y) = \lambda_{J0} \cosh^{-1}\left[\frac{y}{\lambda_{J0}}\right], \tag{4.17}$$

$$j(y) = -2 \cosh^{-2}\left[\frac{y}{\lambda_{J0}}\right] \sinh\left[\frac{y}{\lambda_{J0}}\right]. \tag{4.18}$$

The quantities $H(y)$, $J(y)$ and $f(y)$ are given by (4.11), (4.14) and (4.15), respectively, with $h(y)$ taken from (4.17).

By inserting (4.16) into (3.37) with $H = 0$, we obtain the vortex self-energy:

$$E_v = 8r(T) \frac{\epsilon}{\sqrt{2 + \epsilon^2}}. \tag{4.19}$$

The vortex flux, according to (3.36), is

$$\Phi = \pi\frac{\epsilon^2}{2+\epsilon^2},$$

and the lower critical field, by (3.38), is

$$H_{c1} = \frac{2}{\pi}H_s = \frac{2}{\pi}\frac{\sqrt{2+\epsilon^2}}{\epsilon}. \tag{4.20}$$

Thus, for $\epsilon \ll 1$, the vortex flux $\Phi \ll \Phi_0 = \pi$, and the lower critical field (4.20) is much larger than the corresponding field of a single junction with thick electrodes $H_{c1} = \frac{2}{\pi}\sqrt{\frac{p}{2\lambda}}$, in agreement with Ref.[5].

3. The vortex solution for small Josephson screening

When the screening by Josephson currents is negligibly small, i.e., *(i)* for $L \ll \lambda_{J0}$ and arbitrary H, or *(ii)* for $H_s \ll H \ll H_{c2}$ and arbitrary L, equations (4.5)-(4.9) become

$$\phi(y) = \pi N_v + 2Hy$$

$$-\frac{(-1)^{N_v}}{4\lambda_{J0}^2 H^2}\left[\sin(2Hy) - 2Hy\cos(HW)\right], \tag{4.21}$$

where $N_v = \left[\frac{HW}{\pi}\right]$. Equations of the type (4.21) were first obtained for the infinite layered superconductor by means of a perturbation theory.[1,2] As can be seen from (4.21), the overlapping of states with different N_v now vanishes, and the period of the vortex structure is given by $P = \frac{\pi}{H}$, in full agreement with the general results of sections IIIA and IIIC.

B. A double-junction stack ($N = 3$)

Owing to the symmetry (2.21), $\phi_1(y) = \phi_2(y) \equiv \phi(y)$, and a double-junction stack is described by the single SG equation (4.1) with the single Josephson length

$$\lambda_{J0} = \frac{\epsilon}{\sqrt{1+\epsilon^2}}. \tag{4.22}$$

An exact solution to the boundary value problems (3.22) and (3.29) is again given by Eqs. (4.5)-(4.9), where now N_v should be interpreted as the number of vortex planes. Thus, practically all the formulas of the previous subsection (with corresponding redefinition of the physical quantities) hold for the double-junction stack too. Note that

$$\phi_1(y) = \phi_2(y) \equiv \phi(y), \quad H_1(y) = H_2(y) \equiv H(y), \quad h_1(y) = h_2(y) \equiv h(y),$$

$$j_{1,0}(y) = j_{2,1}(y) \equiv j(y), \quad J_0(y) = J_2(y) \equiv J(y), \quad f_0(y) = f_2(y) \equiv f(y),$$

$$J_1(y) = 0, \quad f_1(y) = 1 - \frac{r(T)}{\lambda_{J0}^2}h(y), \quad H_1 = H_2 = \frac{H}{1+\epsilon^2}.$$

According to (3.25), the superheating (penetration) field is

$$H_s = \lambda_{J0}^{-1} = \frac{\sqrt{1+\epsilon^2}}{\epsilon},$$

which is smaller than the corresponding single-junction value (4.4), in agreement with Ref.[11].

The vortex-plane self-energy for the interval $(-\infty, +\infty)$ is

$$E_v = 16r(T)\lambda_{J0} = 16r(T)\frac{\epsilon}{\sqrt{1+\epsilon^2}}, \tag{4.23}$$

and the flux is

$$\Phi = 2\pi\frac{\epsilon^2}{1+\epsilon^2},$$

which immediately leads to the lower critical field:

$$H_{c1} = \frac{2}{\pi}H_s = \frac{2}{\pi}\frac{\sqrt{1+\epsilon^2}}{\epsilon}.$$

As can be seen by comparing (4.23) with the single-junction expression (4.19), the energy per vortex in the double-junction stack is higher. The vortex-plane solutions are presented schematically in Fig. 2.

V. DISCUSSION

Within the framework of standard methods of the theory of differential equations, we have obtained a complete mathematical description of the Meissner effect and the vortex structure in periodic thin-layer $(N-1)$-junction stacks ($2 \leq N < \infty$). The results of our analytical analysis of the coupled static SG equations (3.1), summarized in **Lemmas 1**,**2** and the **Theorem**, should provide a basis for any further analytical or numerical study of these equations, not necessarily restricted to the field of superconductivity.

By proving the absence of single-vortex solutions to (3.1) for $H > 0$ and specifying the actual domain of their existence ($H = 0$, $L = \infty$), we have clarified a wide-spread misunderstanding:[17,18,15,19] The previous estimates of H_{c1} turn out to be irrelevant because of absolute thermodynamic instability of single-vortex configurations. In full agreement with our study of infinite ($N = \infty$) layered superconductors,[1,2] we have shown that the only true topological (vortex) configurations that survive at $H > 0$ are the vortex planes. Being a natural generalization of ordinary Josephson vortices in a single junction ($N = 2$), the vortex-plane solutions for $3 \leq N$ inherit such properties of the former as periodicity along the layers [Eq. (3.15)] and the overlapping of states with different topological numbers N_v [Eq. (3.30)]. A unified mathematical approach of this paper, valid for $2 \leq N < \infty$, allowed us to derive an exact expression [Eq. (3.25)] for the superheating (penetration) field H_s as an explicit function of N. Within the framework of the same unified approach, we have obtained a new exact solution to the single SG equation, Eqs. (4.5)-(4.9), valid for any $H \geq 0$ and $L > 0$. This solution refutes the assertions[30,7] that Josephson vortices "do not form" in single junctions with small L.

Our closed-form analytical expression for the Meissner field, Eq. (3.27), clearly illustrates an important feature of the Meissner effect in $(N-1)$-junction stacks, not noticed in previous theoretical publications, namely the existence of $\left[\frac{N}{2}\right]$ different Josephson lengths λ_{Jk} ($k = 0, 1, \ldots, \left[\frac{N}{2}\right] - 1$). This result may prove to be useful in view of the current experimental efforts[31,32] to verify the interlayer tunneling model of high-T_c superconductivity[33] by measuring the c-axis penetration depth. [The penetration of the parallel magnetic field with a distribution of length scales has been recently observed[34] in the organic layered superconductor κ-(BEDT-TTF)$_2$Cu(NCS)$_2$.]

Finally, the vortex-plane solutions of our paper provide a natural explanation for the observed coherent Josephson-vortex configurations in artificial double-junction stacks[12] and weakly-coupled multilayers[13] in the presence of a static parallel external field $H > 0$. The observation of isolated interlayer vortices in the high-T_c superconductor $Tl_2Ba_2CuO_{6+\delta}$[31] and in κ-(BEDT-TTF)$_2$Cu(NCS)$_2$[34] can be explained by the fact that in these experiments the external field was set equal to zero ($H = 0$). In this situation, isolated Josephson vortices can be pinned by structural defects, because their self-energy is lower and the spatial extension along the c-axis is smaller than those of the vortex planes at $H = 0$. We hope that our results will stimulate further experimental investigations.

APPENDIX A: THE SOLUTION OF THE FINITE DIFFERENCE EQUATION FOR $H_N(Y)$

Equations (2.17) can be regarded as a nonhomogeneous finite difference equation for $H_n(y)$ with respect to the layer index n, subject to boundary conditions (2.8). According to general theory of such equations,[35] its solution can be represented in the form

$$H_n(y) = h_n(y) + H_n, \tag{A1}$$

where

$$h_n(y) = \frac{\epsilon^2}{2} \sum_{m=1}^{N-1} G(n,m) \frac{d\phi_m(y)}{dy} \tag{A2}$$

is the particular solution of (2.17) satisfying the boundary conditions

$$h_0(y) = h_N(y) = 0, \tag{A3}$$

and

$$H_n = \frac{H\left(\mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n}\right)}{\mu^{-N} - \mu^N}, \tag{A4}$$

$$\mu = 1 + \frac{\epsilon^2}{2} - \epsilon\sqrt{1 + \frac{\epsilon^2}{4}},\ (0 < \mu < 1), \tag{A5}$$

is the solution of the homogeneous form of (2.17) (with the zero right-hand side) meeting the boundary conditions

$$H_0 = H_N = H. \tag{A6}$$

The quantities $G(n,m)$ in (A2) are elements of a $(N-1) \times (N-1)$ matrix $\mathbf{G}(1 \le n, m \le N-1)$. They obey the nonhomogeneous finite difference equation

$$\left(2 + \epsilon^2\right) G(n,m) - G(n+1,m) - G(n-1,m) = \delta_{n,m} \tag{A7}$$

($\delta_{n,m}$ is the Kronecker index) with the boundary conditions

$$G(0,m) = G(N,m) = 0. \tag{A8}$$

The explicit form of $G(n,m)$ is

$$G(n,m) = \frac{1}{2\epsilon\sqrt{1 + \frac{\epsilon^2}{4}}} \left[\mu^{|n-m|} - \frac{\mu^n\left(\mu^{m-N} - \mu^{N-m}\right) + \mu^{N-n}\left(\mu^{-m} - \mu^m\right)}{\mu^{-N} - \mu^N}\right]. \tag{A9}$$

The following properties of $G(n,m)$ can be easily verified using (A7) and (A9):

$$G(n,m) = G(m,n), \tag{A10}$$

$$G(n, N-m) = G(N-n, m), \tag{A11}$$

$$G(n,m) > 0 \text{ for any } 1 \le n, m \le N-1, \tag{A12}$$

$$\sum_{m=1}^{N-1} G(n,m) = \frac{1}{\epsilon^2}\left[1 - G(n,1) - G(n,N-1)\right]$$

$$= \frac{1}{\epsilon^2}\left[1 - \frac{\mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n}}{\mu^{-N} - \mu^N}\right], \quad 1 \le n \le N-1, \tag{A13}$$

$$\sum_{n=1}^{N-1}\sum_{m=1}^{N-1} G(n,m) = \frac{1}{\epsilon^2}\left[N - 1 - \frac{2\sqrt{1 + \frac{\epsilon^2}{4}} - \epsilon}{\epsilon} \frac{1 - \mu^{N-1}}{1 + \mu^N}\right] \equiv \frac{N-1}{\epsilon^2 H_s^2}. \tag{A14}$$

According to (A7), the inverse of the matrix $\mathbf{G}(n,m)$ is a Jacobian matrix[36] $\mathbf{G^{-1}}(n,m)$ with elements $G^{-1}(n,n) = 2 + \epsilon^2$ $(n = 1, 2, \ldots, N-1)$, $G^{-1}(n+1,n) = G^{-1}(n,n+1) = -1$ $(n = 1, 2, \ldots, N-2)$, and $G^{-1}(n,m) = 0$ for $|n-m| > 1$. Hence all the eigenvalues e_j of $\mathbf{G}(n,m)$ are positive and given by

$$e_j = \frac{\lambda_j^2}{\epsilon^2}, \qquad \lambda_j = \frac{\epsilon}{\sqrt{2+\epsilon^2 - 2\cos\frac{\pi j}{N}}}, \qquad i = 1, 2, \ldots, N-1, \tag{A15}$$

with the corresponding eigenvectors

$$u_{ij} = \sqrt{\frac{2}{N}} \sin\frac{\pi i j}{N}, \qquad i, j = 1, 2, \ldots, N-1. \tag{A16}$$

The quantities λ_j determine the characteristic length scales of the stack with N S-layers.

Note that for an infinite layered superconductor, the analog of the matrix $\mathbf{G}(n,m)$ is $\mathbf{G}_\infty(n,m)$, determined by the matrix elements

$$G_\infty(n,m) = \frac{\mu^{|n-m|}}{2\epsilon\sqrt{1+\frac{\epsilon^2}{4}}}, \quad -\infty < n, m < +\infty, \tag{A17}$$

that satisfy the summation rule

$$\sum_{m=1}^{N-1} G_\infty(n,m) = \frac{1}{\epsilon^2}.$$

APPENDIX B: PROOF OF LEMMA 1

By introducing new functions

$$\psi_1(y) = \phi_1(y), \psi_2(y) = \phi_2(y), \ldots, \psi_{N-1}(y) = \phi_{N-1}(y),$$

$$\psi_N(y) = \frac{d\phi_1(y)}{dy}, \psi_{N+1}(y) = \frac{d\phi_2(y)}{dy}, \ldots, \psi_{2N-2}(y) = \frac{d\phi_{N-1}(y)}{dy}, \tag{B1}$$

we convert (3.1) into an equivalent normal system of $2N-2$ first-order equations

$$\frac{d\psi_i(y)}{dy} = F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right), \quad 1 \le i \le 2N-2, \tag{B2}$$

$$F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right) \equiv \psi_{i+N-1}, \quad 1 \le i \le N-1,$$

$$F_N\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right) \equiv \frac{1}{\epsilon^2}\left[\left(2+\epsilon^2\right)\sin\psi_1 - \sin\psi_2\right],$$

$$F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right) \equiv \frac{1}{\epsilon^2}\left[\left(2+\epsilon^2\right)\sin\psi_i - \sin\psi_{i-1} - \sin\psi_{i+1}\right], \quad N+1 \le i \le N-3,$$

$$F_{2N-2}\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right) \equiv \frac{1}{\epsilon^2}\left[\left(2+\epsilon^2\right)\sin\psi_{N-1} - \sin\psi_{N-2}\right],$$

subject to initial conditions

$$\psi_i(y_0) = \alpha_i, \quad 1 \le i \le N-1,$$

$$\psi_i(y_0) = \beta_{i-N+1}, \quad N \le i \le 2N-2. \tag{B3}$$

To prove the statement of **Lemma 1**, it is sufficient to observe that all $F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right)$ are continuous functions of their arguments for $y \in (-\infty, +\infty)$ and $\psi_k \in (-\infty, +\infty)$ $(1 \le k \le 2N-2)$. Moreover, their partial derivatives with respect to ψ_k satisfy the relation

$$\left| \frac{\partial F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right)}{\partial \psi_k} \right| \leq \frac{4+\epsilon^2}{\epsilon^2} \tag{B4}$$

for $y \in (-\infty, +\infty)$ and $\psi_k \in (-\infty, +\infty)$ $(1 \leq i, k \leq 2N-2)$. Thus, the Lipschitz conditions with respect to ψ_k are met for $y \in (-\infty, +\infty)$ and $\psi_k \in (-\infty, +\infty)$ $(1 \leq k \leq 2N-2)$, which immediately guarantees[8] the existence and uniqueness of a solution to (B2), satisfying arbitrary initial conditions (B3), in an arbitrary interval $I = [L_1, L_2]$ such that $y_0 \in I$. Continuous dependence of the solution on initial data is a result of continuous dependence of $F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right)$ on their arguments and of the condition (B4). Infinite differentiability of the solution automatically follows from infinite differentiability of $F_i\left(\psi_1, \psi_2, \ldots, \psi_{2N-2}\right)$ with respect to their arguments.

APPENDIX C: PROOF OF THE THEOREM

Owing to the property (3.3), without loss of generality, we can consider the initial value problem (3.14) for $y = y_0 \equiv 0$. To prove the Theorem, we have to show that there exists a solution to (3.1) in $(-\infty, +\infty)$ with the properties (3.15)-(3.19) that at $y = y_0 \equiv 0$ satisfies (3.14). By **Lemma1**, this solution will represent the sought unique solution to the initial value problem (3.14).

Consider an arbitrary finite interval $I = \left[-\frac{P}{2}, \frac{P}{2}\right]$. We start with the Meissner boundary value problem in the interval I:

$$\frac{d\phi_n}{dy}\left(\pm\frac{P}{2}\right) = 2\tilde{H} > 0, \qquad \phi_n(0) = 0, \qquad 1 \leq n \leq N-1. \tag{C1}$$

Up to a certain $\tilde{H} = \tilde{H}_s$, the problem (C1) has a unique solution with the properties

$$\phi_n(y) = -\phi_n(-y), \qquad y \in I, \qquad 1 \leq n \leq N-1, \tag{C2}$$

$$\frac{d\phi_n}{dy}(y) > 0, \qquad y \in I, \qquad 1 \leq n \leq N-1, \tag{C3}$$

$$-\pi \leq \phi_n\left(-\frac{P}{2}\right) < 0, \qquad 0 < \phi_n\left(\frac{P}{2}\right) \leq \pi, \qquad 1 \leq n \leq N-1, \tag{C4}$$

where $\tilde{H}_s$ is determined by the conditions

$$\phi_n\left(\pm\frac{P}{2}\right) = \pm\pi, \qquad 1 \leq n \leq N-1, \tag{C5}$$

and satisfies the inequality $\tilde{H}_s > H_s$, with H_s defined via (A14). (See section IIIB.)

Now we will prove that *all* $\frac{d\phi_n}{dy}(0)$, where ϕ_n are the solution to (C1), for $\tilde{H} \to \tilde{H}_s$ tend to the *same* limiting value

$$\frac{d\phi_n}{dy}(0) = 2\sqrt{\tilde{H}_s^2 - H_s^2}, \qquad 1 \leq n \leq N-1. \tag{C6}$$

Indeed, relations (C6) satisfy (3.5) at $y = 0$ with (3.6), where the substitution $H \to \tilde{H}_s$, $L \to \frac{P}{2}$ has been made. However, it is still necessary to show that the limiting value (C6) is uniquely determined by the solution to (C1). To this end, we consider the relation

$$4(N-1)\frac{\tilde{H}}{H_s^2} - \epsilon^2 \sum_{n=1}^{N-1}\sum_{m=1}^{N-1} G(n,m)\frac{d\beta_n}{d\tilde{H}}\beta_m = \sum_{n=1}^{N-1} \sin\phi_n\left(\frac{P}{2}\right)\frac{d\phi_n\left(\frac{P}{2}\right)}{d\tilde{H}} \geq 0, \tag{C7}$$

where $\beta_n \equiv \frac{d\phi_n}{dy}(0)$. [Relation (C6) is obtained from (3.5), (3.6) by the substitution $H \to \tilde{H}$, $L \to \frac{P}{2}$ and differentiation with respect to $\tilde{H}$.] Notice that

$$\beta_n < 2\tilde{H}, \qquad 1 \leq n \leq N-1, \tag{C8}$$

in the whole range $0 < \tilde{H} \leq \tilde{H}_s$, because all $\frac{d\phi_n}{dy}$ have minima at $y = 0$. [See (3.4).] With increasing $\tilde{H}$ from zero to $\tilde{H}_s$, all $\phi_n\left(\frac{P}{2}\right)$ increase monotonously from zero to π. Thus, the difference on the left-hand side of (C6) first increases (for $0 < \tilde{H} \ll H_s$) and then decreases (for $H_s \leq \tilde{H} \leq \tilde{H}_s$). It means that the dependence $\beta_n(\tilde{H})$ changes from a linear one for $0 < \tilde{H} \ll H_s$ [i.e., $\beta_n(\tilde{H}) = \bar{b}_n\tilde{H}$, $0 < \bar{b}_n < 2$, $1 \leq n \leq N-1$] to a nonlinear one for $H_s \leq \tilde{H} \leq \tilde{H}_s$, with

$$\frac{d\beta_n}{d\tilde{H}} > 2, \qquad 1 \leq n \leq N-1. \tag{C9}$$

At $\tilde{H} = \tilde{H}_s$ the right-hand side of (C6) is equal to zero, and we have

$$\frac{d\beta_n}{d\tilde{H}_s}\beta_m = b_n b_m \tilde{H}_s, \tag{C10}$$

where b_n satisfy the conditions

$$\epsilon^2 \sum_{n=1}^{N-1}\sum_{m=1}^{N-1} G(n,m) b_n b_m = \frac{4\,(N-1)}{H_s^2}, \tag{C11}$$

$$0 < b_{\min} \leq b_n \leq b_{\max}, \qquad b_{\min} = \min_n \{b_n\} \leq 2, \qquad b_{\max} = \max_n \{b_n\} \geq 2. \tag{C12}$$

The integration of (C10) for $n = m$ under the condition $\beta_n(H_s) = 0$ yields

$$\beta_n\left(\tilde{H}_s\right) = b_n\sqrt{\tilde{H}_s^2 - H_s^2}. \tag{C13}$$

With the help of (C8) and (C9), taken at $\tilde{H} = \tilde{H}_s$, we establish the following inequalities for $b_{\min}$ and $b_{\max}$:

$$2\sqrt{1 - \frac{H_s^2}{\tilde{H}_s^2}} < b_{\min} \leq 2, \qquad 2 \leq b_{\max} < \frac{2}{\sqrt{1 - \frac{H_s^2}{\tilde{H}_s^2}}}. \tag{C14}$$

Inequalities (C14) must hold for any $\tilde{H}_s > H_s$. Proceeding to the limit $\frac{H_s^2}{\tilde{H}_s^2} \to 0$, we get $b_{\min} = b_{\max} = 2$, hence the relations (C6).

The Meissner solution at $\tilde{H} = \tilde{H}_s$ can be uniquely continued from the interval I into the whole interval $(-\infty, +\infty)$.[8] The solution ϕ_n $(1 \leq n \leq N-1)$ thus obtained possesses all the required properties (3.14)-(3.19), where $y_0 \equiv 0$, and $H \equiv \sqrt{\tilde{H}_s^2 - H_s^2}$. To show this, it is sufficient to prove the property (3.15). [The properties (3.14), (3.17) and (3.18) are obvious. The property (3.16) results from the differentiation of (3.15). The property (3.19) is obtained from (3.5), (3.6) by integration.]

Consider a set of functions

$$\bar{\phi}_n(y) = \phi_n(y + P) - 2\pi, \qquad 1 \leq n \leq N-1, \tag{C15}$$

where $\phi_n(y)$ $(1 \leq n \leq N-1)$ are the continuation of the Meissner solution at $\tilde{H} = \tilde{H}_s$ into the interval $(-\infty, +\infty)$. Owing to the properties (3.2), (3.3), the functions (C15) also represent a solution to (3.1) in the interval $(-\infty, +\infty)$. At $y = -\frac{P}{2}$, the solution $\bar{\phi}_n$ $(1 \leq n \leq N-1)$ satisfies the same initial conditions as the solution ϕ_n $(1 \leq n \leq N-1)$. By **Lemma 1**, the latter means that $\bar{\phi}_n \equiv \phi_n$ $(1 \leq n \leq N-1)$ in the interval $(-\infty, +\infty)$, i.e.,

$$\phi_n(y + P) - 2\pi = \phi_n(y), \qquad 1 \leq n \leq N-1, \tag{C16}$$

which accomplishes the proof of the **Theorem**. Note that in this proof we have not employed the symmetry relations (2.21) resulting from the boundary conditions (2.8).

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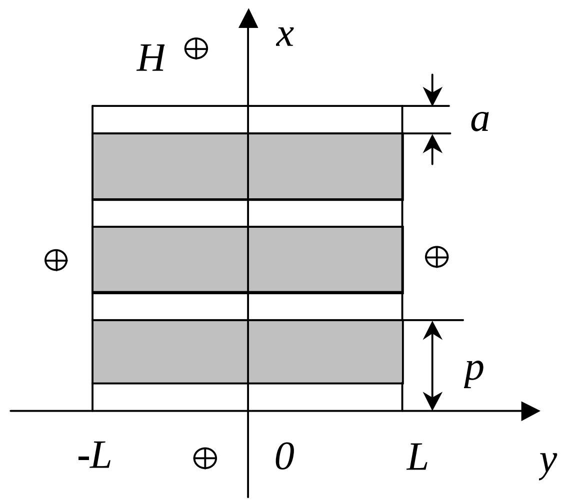
FIGURE CAPTIONS

1. Fig. 1. The geometry of the problem (schematically). Here $N = 4$, $a \ll p$ (a is the S-layer thickness), $2L = W$, $H > 0$.

2. Fig. 2. The vortex-plane solutions in a double-junction stack (schematically): a) the state with a single vortex plane ($N_v = 1$); b) the state with two vortex planes ($N_v = 2$). The black dots conventionally denote the positions of Josephson vortices, and the arrows show the distribution of intralayer and Josephson currents.



H
x
a
p
-L
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L
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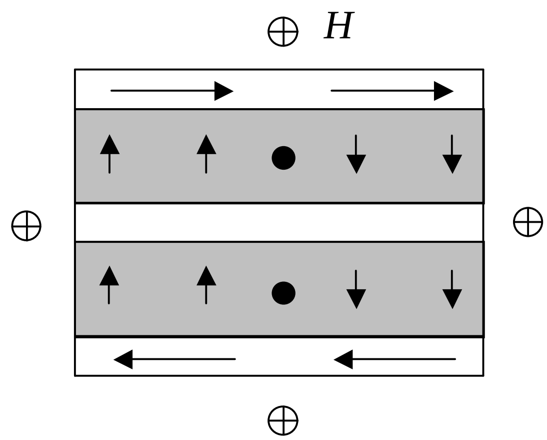
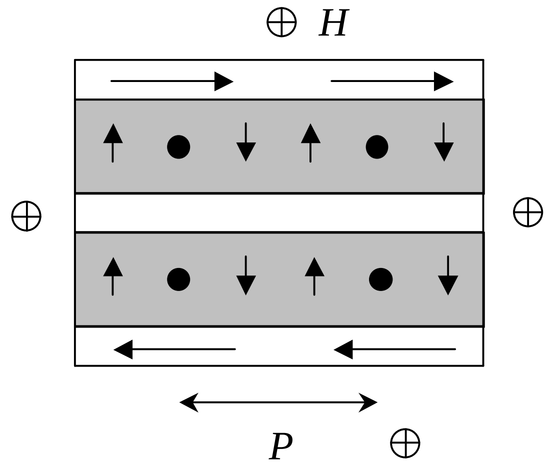

H


H
P